Confluent Term Rewriting for Only-knowing Logics

Espen H. LIAN [a,1], Einar Broch JOHNSEN [a] and Arild WAALER [a]
[a] *Department of Informatics, University of Oslo, Norway*

Abstract. Combining term rewriting and modal logics, this paper addresses confluence and termination of rewrite systems introduced for only-knowing logics. The rewrite systems contain a rule scheme that gives rise to an infinite number of critical pairs, hence we cannot check the joinability of every critical pair directly, in order to establish local confluence. We investigate conditions that are sufficient for confluence and identify a set of rewrite rules that satisfy these conditions; however, the general confluence result makes it easier to check confluence also of stronger systems should one want additional rules. The results provide a firm logical basis for implementation of procedures that compute autoepistemic expansions.

Keywords. Nonmonotonic reasoning, Term rewriting, Only-knowing logics

1. Introduction

One way of motivating only-knowing logics is through their intended application to nonmonotonic reasoning. Although only-knowing logics have successfully been used to represent autoepistemic and default logics [14,11,21,15], the only-knowing logic addressed in this paper primarily relates to propositional autoepistemic logic, the simplest case in point. Let us, before we motivate and explain the contribution of this paper, briefly summarize autoepistemic logic and the way it is reflected in only-knowing logic.

The language of autoepistemic propositional logic is a modal language with a single belief modality B. The central notion of a stable set is defined as follows: A set Γ of autoepistemic formulae is *stable* if it is closed under propositional logic, $B\psi \in \Gamma$ for each $\psi \in \Gamma$ and $\neg B\psi \in \Gamma$ for each $\psi \notin \Gamma$. The subset of Γ consisting only of propositional formulae (i.e. formulae without modalities) is the kernel of Γ. This kernel is unique; moreover, each deductively closed set of propositional formulae is the kernel of a unique stable set.

Given an autoepistemic formula φ, the autoepistemic consequence relation determines the stable expansions of φ, i.e. stable sets that entail it and that satisfy a specific fixpoint equation. Despite the non-constructive nature of the consequence relation, there are simple algorithms [1,5,14] that for each autoepistemic formula φ compute a set of propositional formulae $\varphi_1, \ldots, \varphi_n$ such that $\mathrm{Th}(\varphi_1), \ldots, \mathrm{Th}(\varphi_n)$ are the kernels of all and only stable expansions of φ, where $\mathrm{Th}(\varphi_i)$ denotes the set of propositional consequences of φ_i. We illustrate a simple algorithm on the so-called *Nixon Diamond*, a well-known example illustrating conflicting defaults.

[1] Corresponding Author: Espen H. Lian, E-mail: elian@ifi.uio.no

Example 1. *Assume a knowledge base* KB*, a formula of propositional logic, which entails that Nixon is both a quaker and a republican, formalized as* $q \wedge r$*. "Quakers are normally pacifists" is, when applied to Nixon, expressed as a formula* $\delta_q = (Bq \wedge \neg B\neg p) \supset p$*. It intuitively reads "If I believe that Nixon is a quaker and it is consistent with my beliefs that he is a pacifist, then he is a pacifist." "Republicans are normally not pacifists" applied to Nixon is expressed as* $\delta_r = (Br \wedge \neg Bp) \supset \neg p$*. We want to find the stable expansions of the formula* $\varphi = \text{KB} \wedge \delta_q \wedge \delta_r$*.*

The algorithm first identifies the modal atoms that occur as subformulae of φ*:* Bq*,* Br*,* Bp*,* $B\neg p$*. There are potentially 16 different ways in which these may be valuated, each of which is treated separately. Consider the valuation that maps* Bq*,* Br *and* Bp *to true and* $B\neg p$ *to false. Take the modal atoms that are valuated to true and determine the propositional consequences of these three and* φ *taken together. The result is Th*$(\text{KB} \wedge p)$*, since the* δ_q *default yields* p*. This is potentially the kernel of a stable expansion of* φ*. To verify that Th*$(\text{KB} \wedge p)$ *is indeed a stable expansion, we must check that* $\text{KB} \wedge p$ *entails propositions which are believed to hold (i.e.* $q \wedge r \wedge p$*) by the assumed valuation and that it does not entail any proposition that is not believed to hold (i.e.* $\neg p$*). Since both these tests go through, Th*$(\text{KB} \wedge p)$ *is the kernel of a stable expansion of* φ*. It is easy to check that Th*$(\text{KB} \wedge \neg p)$ *is the kernel of another stable expansion of* φ*, and that there are no other stable expansions.*

The algorithm for autoepistemic logic illustrated above manipulates a quite complex construction in the meta-language; enumerating potential valuations, generating consequences and checking for consistencies. It is precisely this reasoning at the meta-level underlying the algorithm, that only-knowing logics can accommodate. Only-knowing logics do not only represent this pattern of reasoning *at the object level* of the logic, they can also replace the construction in the algorithm with a *calculus*.

Example 2. *In only-knowing logic the Nixon Diamond can be represented by the formula* $O\varphi$*, expressing that* φ *is "all I know;" the formula conveys both that "I believe that* φ *holds" and that "whatever I believe is a consequence of* φ*" (the latter statement is formalized by means of a co-belief operator* C *specifically addressed in [21]). By the rules of the logic, the equivalence* $O\varphi \equiv O(\text{KB} \wedge p) \vee O(\text{KB} \wedge \neg p)$ *is a theorem. Note that the disjuncts to the right in the equivalence correspond directly to the stable expansions. The right hand side of the equivalence can be determined from the left hand side in many ways; the procedure addressed in this paper first applies a so-called expand rule and then collapse rules and structural rules. The expand rule can be viewed as a rule that successively builds up valuations of the sort addressed in Example 1. The expand rule will, e.g., map the modal atom* Bq *in* $O\varphi$ *to either* $\top$ *or* $\bot$*:*

$$O\varphi \rightarrow (O\varphi\langle Bq/\top\rangle \wedge Bq) \vee (O\varphi\langle Bq/\bot\rangle \wedge \neg Bq).$$

Here $O\varphi\langle Bq/v\rangle$ *evaluates to* $O(\text{KB} \wedge ((v \wedge \neg B\neg p) \supset p) \wedge \delta_r)$*. We can apply a distribution rule, expand wrt. the remaining modal atoms, and simplify the result to get a formula on DNF, in which one of the 16 disjuncts (corresponding to the potential kernels of stable expansions identified by the procedure in Example 1) is:*

$$O(\text{KB} \wedge p) \wedge Bq \wedge \neg B\neg p \wedge Bp \wedge Br.$$

A consistency check is now implemented using collapse rules (cf. Section 5), in this case leading to $O(\text{KB} \wedge p)$; the collapse rules reduce an O-formula and a modal atom to either the O-formula itself or to $\bot$ (most of the disjuncts in this example reduce to $\bot$). In the end we get a formula with two consistent disjuncts: $O(\text{KB} \wedge p) \vee O(\text{KB} \wedge \neg p)$.

The Modal Reduction Theorem [21] states that for each formula φ, there exists propositional formulae $\psi_1, \ldots, \psi_n$ such that $O\varphi \equiv O\psi_1 \vee \cdots \vee O\psi_n$ is provable, where each $O\psi_i$ has an essentially unique model that corresponds to a stable expansion (it forms a complete theory over the subjective, i.e. completely modalized, fragment of the language). Hence, the theorem guarantees that one can reduce $O\varphi$, representing an autoepistemic theory, to a formula of a form which directly exhibits its models. Note that the validity of the equivalence in the Modal Reduction Theorem shows that only-knowing logic is in itself strong enough to accommodate the reasoning used to determine stable expansions in autoepistemic logic. Also note that only-knowing logics are, in contrast to autoepistemic logic, in themselves monotonic; it is the arguments to O-modalities that exhibit nonmonotonic behaviour. Being normal systems of modal logic they have a standard Kripke semantics that in particular provides autoepistemic logic with increased conceptual clarity compared to its original fixpoint definition of expansions, not least because only-knowing systems sharply separate object-language features from meta-concepts.

Whereas the Modal Reduction Theorem gives an abstract characterization of the expressivity of only-knowing logic, the aim of this paper is to provide a bridge to the design and implementation of procedures for computing expansions. In [21] the Modal Reduction Theorem is proved using a set of equivalence-preserving rewrite rules, providing an easy-to-use calculus for reducing $O\varphi$ to $O\psi_1 \vee \cdots \vee O\psi_n$; we shall refer to the latter as the *canonical form* of the former. Compared to more high-level (pseudo-code) algorithmic specifications, like the one in Example 1, rewrite systems are easier to reason about and may be implemented more directly, for instance using the rewriting logic tool Maude [3]. To pave the way for such principled and high-level approaches to implementation we shall in this paper focus on *conditions for the design of confluent rewrite calculi for only-knowing logics*.

The algorithm in Example 1 is reminiscent of a truth table method for checking propositional satisfiability: list all potential models and check each of them in separation. Clearly, one can do much better, but using the framework of autoepistemic logic the correctness of the algorithm must be checked for each optimization. Adding rules to a confluent system is easier to deal with; all we have to do is to check that confluence is preserved. Once this is done, we get the flexibility to modify search procedures for free. The confluence results in this paper provide a firm logical basis for the implementation of procedures that compute autoepistemic expansions which will, we believe, facilitate the study and comparison of different search strategies. No results like the ones presented in this paper have, to our knowledge, previously been established for a logic that supports nonmonotonic patterns of reasoning.

For standard systems of rewriting modulo an equivalence relation, confluence is proved using the following line of argument:

(i) Local confluence follows from joinability of all critical pairs.
(ii) Confluence follows from local confluence, local coherence with the equivalence relation, and termination.

Hence, one has to check joinability of critical pairs, local coherence and termination; in many cases these follow straightforwardly using standard techniques. The expand rule, illustrated in Example 2, is in fact a rule *scheme* that gives rise to an infinite number of critical pairs, hence we need a systematic method to establish the joinability of every critical pair.

The collapse rules are expensive as they are preconditioned by SAT (propositional satisfiability) tests, a fact which efficient strategies must deal with. The proof of the Modal Reduction Theorem based on formula rewriting uses just the set of rewrite rules that is needed to establish the theorem. The canonical form of an only-knowing formula is derived by first generating a large DNF formula in which each disjunct corresponds to a potential model, and then reducing each disjunct either to $\bot$ or to a satisfiable only-knowing formula. This rule set consists of modal rules and two propositional structural rules: a distribution rule and a contradiction rule. From a computational perspective it is clearly desirable to have more simplification rules at hand, to avoid having to generate the large DNF formula upfront.

What we need is hence not just one confluent system for only-knowing logic, but a systematic method, that can be used to check confluence of rewrite systems with the expand rule. A result of this kind is not straightforward to establish. In Section 4 we suggest a solution by introducing a system R_{Min} and a natural condition that establishes that critical pairs stemming from the use of the expand rule are joinable. In Section 5 we prove that the original rewrite system in [21] equipped with some extra structural rules is indeed confluent.

2. Only-knowing Logic

The only-knowing language we address in this paper slightly generalizes the language discussed in Section 1 in that it is parameterized by an index set I partially ordered by a relation $\preccurlyeq$. Intuitively each index in I denotes a confidence layer of an agent. Relative to each confidence layer, there are the modal operators B_k (belief), C_k (co-belief) and O_k (exact belief). Atomic formula consist of the truth constants $\top$ and $\bot$, and a set $\{p_i\}_{i\geqslant 1}$ of *propositional letters*; we will sometimes write p, q and r for p_1, p_2, and p_3. Formulae are constructed as follows.

$$\varphi, \psi \longrightarrow \top \mid \bot \mid p_i \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid B_k\varphi \mid C_k\varphi \mid O_k\varphi$$

for all $k \in I$, where I is a finite index set. A formula of the form $B_k\varphi$ or $C_k\varphi$ is called a *modal atom*. A modal *literal* is a modal atom or its negation. We say that a formula with an occurrence of O_k is *tainted*; the reason for discriminating against tainted formulae has to do with confluence issues. A formula is *subjective* if every propositional letter is within the scope of a modal operator. A subjective formula is *prime* if it contains no nested modalities. A formula without any occurrence of a modal operator is called *objective*. A formula is *constant* if it is a Boolean combination of the truth constants. Note that a constant formula is both subjective and objective; subjective because it contains no propositional letters and objective because it contains no modalities. The sets $\mathrm{mod}(\varphi)$ and $\mathrm{pmod}(\varphi)$ consist of the modal atoms and prime modal atoms, resp., occurring in φ.

For further motivation, axiomatization and semantics of the only-knowing logic, cf. [21]. Of special interest are formulae of the form $\bigwedge_{k\in I} O_k\varphi_k$, called O_I*-blocks*. O_I-

blocks intuitively represent the belief state of an agent relative to each confidence level in the index set I. Let $\varphi = \bigwedge_{k \in I} O_k \varphi_k$ and $\psi = \bigwedge_{k \in I} O_k \psi_k$ be two prime O_I-blocks. φ is *cumulative* if $i \prec j$ implies that $\varphi_j \supset \varphi_i$ is a tautology, i.e. that $\{\neg\varphi_i, \varphi_j\}$ is unsatisfiable. Cumulativity reflects the intuition behind the partial order $\prec$ on I. φ and ψ are *independent* if there is an $i \in I$ such that $\varphi_i \equiv \psi_i$ is not a tautology.

A fundamental result in only-knowing logic, the Modal Reduction Theorem (MRT) [21], states that any O_I-block φ is equivalent to a disjunction of prime O_I-blocks $\mu_1 \vee \cdots \vee \mu_n$ for some $n \geqslant 0$ such that each μ_k is cumulative. Such a disjunction is on *canonical form* if every pair of disjuncts μ_i and μ_j for $i \neq j$ is independent; observe that the empty disjunction $\bot$ is on canonical form. One goal in this paper is to present a confluent rewrite system where the normal form of O_I-blocks is canonical.

As explained in Section 1, the rewrite procedure is based on substituting truth constants for modal atoms. To this end we need some terminology about substitutions. *Bindings* $a, b, c, \ldots$ are ordered pairs $\langle \beta / v \rangle$ such that β is a prime modal atom and v a truth constant. We view a binding $\langle \beta / v \rangle$ as a function that maps (or *binds*) β to v, that is $\beta\langle \beta / v \rangle = v$ using postfix notation. Bindings are extended to arbitrary formulae in the usual way. A *modal substitution* (henceforth just *substitution*, not to be confused with the ground substitutions in Section 3) is a sequence over some set of bindings; the *empty substitution* is written ε. If σ and τ are sequences, the *concatenation* of σ and τ is denoted $\sigma \cdot \tau$ or simply $\sigma\tau$. We let $\varphi(a\sigma) = (\varphi a)\sigma$ for any substitution $a\sigma$. For the sake of simplicity, we define the complement operation on truth constants: $\overline{\top} = \bot$ and $\overline{\bot} = \top$. Observe that bindings are in general not commutative (and not idempotent): If we let $b = \langle Lv / \overline{v} \rangle$ and $c = \langle L\overline{v} / v \rangle$, then $(LLv)bc = v$, while $(LLv)cb = L\overline{v}$. A set of bindings is *consistent* if no subset is of the form $\{\langle \beta / \top \rangle, \langle \beta / \bot \rangle\}$; a *modal valuation* is a consistent set of bindings. We may view a modal valuation V as a function from prime modal atoms to truth constants, hence we may refer to the *domain* of V: $\mathrm{dom}(V) = \{\beta \mid \langle \beta / v \rangle \in V\}$. A straightforward generalization of Lemma 31 in [21] is:

Lemma 3. *If σ and τ are substitutions over a modal valuation V such that $\varphi\sigma$ and $\varphi\tau$ are without occurrences from* $\mathrm{dom}(V)$*, then $\varphi\sigma = \varphi\tau$.*

We will assume that no formula has nested Os, equivalently that $O_k\varphi$ is a formula iff φ is not tainted, as this complicates the confluence argument unnecessarily. However, this assumption does not decrease expressibility, as any formula $O_k\varphi$ is equivalent to $B_k\varphi \wedge C_k\neg\varphi$, which may occur freely. Hence to express, e.g., O_1O_2p, we must write $O_1(B_2p \wedge C_2\neg p)$.

3. Order-Sorted Term Rewriting

We want to capture the MRT with a rewrite theory, i.e. we want a rule set R such that for any O_I-block φ, there is a μ on canonical form such that $\varphi \twoheadrightarrow_R \mu$. Additionally, we want this rule set to be confluent.

3.1. General Notions

Cf. [2,17] for details on standard definitions and results. Let Σ be a set of function symbols of given sorts – partially ordered by a subsort relation – and arities, X an unbounded

set of variables disjoint from Σ, and denote by T the term algebra over the function symbols Σ and variables X. For a term $\varphi \in T$ and ground substitution θ, we denote by $\theta\varphi$ the application of θ to φ (resulting in a ground term without variables). The set $\mathcal{P}os(\varphi)$ of *positions* and the set $\mathcal{V}ar(\varphi)$ of *variables* in φ are defined in the standard way [2]. Denote by $\varphi|_p$ the subterm of φ at position p and by $\varphi[\psi]_p$ the term φ in which ψ replaces the subterm at position p.

Let R be a set rewrite rules $l \to r$ where $l, r \in T$. We conventionally refer to the *left hand side* l of a rule $l \to r$ as the LHS, and to the *right hand side* r as RHS, and call an instance of a LHS a *redex*. A term φ *reduces* to ψ, written $\varphi \to_R \psi$, if there exists $l \to r \in R$, $p \in \mathcal{P}os(\varphi)$, and $\theta \in Sub(\Sigma)$ such that $\varphi|_p = \theta l$ and $\psi = \varphi[\theta r]_p$. The reflexive transitive closure of $\to_R$ is denoted $\twoheadrightarrow_R$. If the rule set is clear from the context, we simply write $\to$ and $\twoheadrightarrow$. A term φ is on *normal form* if there is no ψ such that $\varphi \to \psi$.

Let AC be a set of equations specifying associativity and commutativity of conjunction and disjunction, and let $\to_{\text{AC}}$ denote the rewrite system derived by orienting each equation $l = r \in$ AC as a rule $l \to r$. We write $\varphi \leftrightarrow_{\text{AC}} \psi$ if $\varphi \to_{\text{AC}} \psi$ or $\psi \to_{\text{AC}} \varphi$, while $=_{\text{AC}}$ denotes the reflexive transitive closure of $\leftrightarrow_{\text{AC}}$. For terms φ and ψ, we write $\varphi \to_{R/\text{AC}} \psi$ if there are terms μ and ν such that $\varphi =_{\text{AC}} \mu \to_R \nu =_{\text{AC}} \psi$. We will simply write R for $R/$AC throughout. Following [4], R is *terminating* if there is no infinite sequence of the form $\varphi_1 \to_R \varphi_2 =_{\text{AC}} \varphi_3 \to_R \varphi_4 =_{\text{AC}} \varphi_5 \to_R \cdots$. We say that φ and ψ are *R-joinable*, denoted $\varphi \downarrow_R \psi$, if there are terms μ and ν such that $\varphi \twoheadrightarrow_R \mu =_{\text{AC}} \nu \twoheadleftarrow_R \psi$. Furthermore, R is *locally confluent* if $\mu \leftarrow_R \varphi \to_R \nu$ implies $\mu \downarrow_R \nu$; R is *locally coherent with AC* if $\mu \leftarrow_R \varphi \leftrightarrow_{\text{AC}} \nu$ implies $\mu \downarrow_R \nu$; and R is *confluent* if $\mu \twoheadleftarrow_R \varphi \twoheadrightarrow_R \nu$ implies $\mu \downarrow_R \nu$.

Lemma 4 ([4,9,10])**.** *R is confluent if it is terminating, locally confluent and locally coherent with AC.*

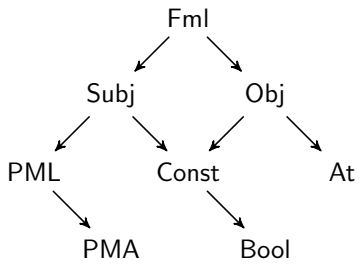
If $l \to r$ and $l' \to r'$ are variable-renamed rules such that $\mathcal{V}ar(l, r) \cap \mathcal{V}ar(l', r') = \emptyset$, p is a non-variable position of l, and θ is a most general unifier of $(l|_p, l')$, then $(\theta r_1, (\theta l)[\theta r_2]_p)$ is called a *critical pair* [2].

Critical Pair Lemma ([2,9,18])**.** *R is locally confluent iff all its critical pairs are joinable.*

3.2. An Order-Sorted Equational Specification

We define a term algebra for the language of only-knowing formulae. It consists of the following sorts: Fml, Subj, Obj, PMA, PML, Const, At and Bool. Let $\mathsf{Bool} = \{\top, \bot\}$, and let At be the set of propositional letters. Terms are constructed as follows. Each sort in the upper diamond-shaped part of the subsort hierarchy (Figure 1) is closed under the Boolean connectives, i.e. $\vee, \wedge : X \times X \Rightarrow X$ and $\neg : X \Rightarrow X$ for $X \in \{\mathsf{Fml}, \mathsf{Subj}, \mathsf{Obj}, \mathsf{Const}\}$. Negating a prime modal atom constructs a prime modal literal: $\neg : \mathsf{PMA} \Rightarrow \mathsf{PML}$. We conventionally use infix notation for the connectives and write the index of the modal operators as subscripts, e.g., $O_k\varphi$ for $O(k, \varphi)$. Prefixing an objective formula with B_k or C_k constructs a prime modal atom: $B, C : I \times \mathsf{Obj} \Rightarrow \mathsf{PMA}$; prefixing a general formula with O_k, B_k or C_k constructs a a subjective formula: $O, B, C : I \times \mathsf{Fml} \Rightarrow \mathsf{Subj}$. It is easy to see that the sorts capture the syntax of the only-knowing logic correctly, e.g., Subj corresponds the set of subjective formulae, as

it comprises prime modal literals and constant formulae, and is closed under boolean connectives. The correspondence between sorts and fragments of the logical language is listed in Figure 1.



Fml	formulae
Subj	subjective formulae
Obj	objective formulae
PML	prime modal literals
PMA	prime modal atoms
Const	constant formulae
At	propositional letters
Bool	truth constants

Figure 1. The subsort hierarchy and intended meaning of the sorts.

4. Confluence of Rewrite Systems with the Expand Rule

As we have seen, the rewriting procedure is based on uniformly substituting a truth constant for every occurrence of some prime modal atom β in a formula φ, using the expand rule, which is actually a rule scheme, and as such gives rise to an infinite number of rules. The *expand rule (scheme)* is defined as

$$O_i\varphi \to_{\mathsf{E}} (O_i\varphi\langle\beta/\top\rangle \wedge \beta) \vee (O_i\varphi\langle\beta/\bot\rangle \wedge \neg\beta) \text{ if } \beta \in \text{pmod}(\varphi).$$

The problem with the expand rule is that it is the source of an infinite number of critical pairs. Hence we need a systematic way to show that critical pairs stemming from the use of the expand rule are joinable. In the rest of this section we investigate the relationship (wrt. joinability) between the expand rule and the *structural rules* in Figure 2. We consider several rule sets with different properties; not all will capture the MRT. All contain the expand rule and additional rules are either a structural rule or the *collapse rules* (Figure 4) introduced in Section 5. The rule set consisting of all of these rules is denoted R_{Only}.

4.1. Permutable Expandability

For $\alpha_1, \ldots, \alpha_n \in \mathsf{PMA}$, let $e(O_i\varphi, \alpha_1, \ldots, \alpha_n)$ be shorthand for

$$\bigvee_{\vec{u}\in\mathsf{Bool}^n} (O_i\varphi\langle\alpha_1/u_1\rangle \cdots \langle\alpha_n/u_n\rangle \wedge \bigwedge_{1\leqslant k\leqslant n} \alpha_k(u_k)),$$

where $\vec{u} = (u_1, \ldots, u_n)$. Using this notation, the expand rule can be written as

$$O_i\varphi \to_{\mathsf{E}} e(O_i\varphi, \alpha) \text{ if } \alpha \in \text{pmod}(\varphi),$$

in which case we say that $O_i\varphi$ is *expanded* wrt. α. A rule set R satisfies *permutable expandability* if $e(O_i\varphi, \alpha) \downarrow_R e(O_i\varphi, \beta)$ for any $\alpha, \beta \in \text{pmod}(\varphi)$; i.e. if $O_i\varphi$ can be

expanded both wrt. α and β, the resulting reducts are joinable. Any rule set containing the expand rule must satisfy this property in order to be confluent. The rule set consisting of just the expand rule, does not satisfy permutable expandability, hence we are interested in stronger systems that do. A minimal (in the sense that we need every rule except A1 to show Lemma 5 below, while A1 is needed to prove Lemma 8) rule set with this property is $R_{\mathsf{Min}} = \{\mathsf{E}, \mathsf{Dt}, \mathsf{Dm}, \mathsf{J}\vee, \mathsf{I}\wedge, \mathsf{Kn}, \mathsf{A1}\}$.

Distribution	$(\psi_1 \vee \psi_2) \wedge \eta \to_{\mathsf{Dt}} (\psi_1 \wedge \eta) \vee (\psi_2 \wedge \eta)$		
Absorption 1	$(\varphi \wedge \beta) \vee (\varphi \wedge \overline{\beta}) \to_{\mathsf{A1}} \varphi$		
Absorption 2	$(\varphi \wedge \beta) \vee (\varphi \wedge \overline{\beta} \wedge \eta) \to_{\mathsf{A2}} (\varphi \wedge \beta) \vee (\varphi \wedge \eta)$		
Absorption 3	$\varphi \vee (\varphi \wedge \eta) \to_{\mathsf{A3}} \varphi$		
Identity of $\wedge$	$\psi \wedge \top \to_{\mathsf{J}\wedge} \psi$	Domination	$\psi \wedge \bot \to_{\mathsf{Dm}} \bot$
Identity of $\vee$	$\psi \vee \bot \to_{\mathsf{J}\vee} \psi$	Contradiction	$\beta \wedge \overline{\beta} \to_{\mathsf{Kn}} \bot$
Idempotency of $\vee$	$\psi \vee \psi \to_{\mathsf{I}\vee} \psi$	Complement of $\top$	$\neg\top \to_{\mathsf{Co}} \bot$
Idempotency of $\wedge$	$\beta \wedge \beta \to_{\mathsf{I}\wedge} \beta$	Complement of $\bot$	$\neg\bot \to_{\mathsf{Co}} \top$

Figure 2. The structural rules. $\psi_1, \psi_2, \psi, \eta \in \mathsf{Subj}$, $\beta \in \mathsf{PML}$, φ is of the form $O\mu$ or $O\mu \wedge \psi$, and η must be untainted. $\overline{\beta}$ denotes the complement of β, i.e. $\overline{L\psi} = \neg L\psi$ and $\overline{\neg L\psi} = L\psi$. A1, A2 and A3 are called *the absorption rules*.

Repeated application of first the expand rule and then the distribution rule aggregates conjunctions of prime modal atoms. Since we rewrite modulo AC, a conjunction of prime modal literals can be viewed as a multiset. With the aid of the structural rules, such a conjunction can be reduced to a form which corresponds to a consistent set: I$\wedge$ removes duplicates and Kn/Dm reduce it to the empty disjunction $\bot$ if inconsistent. The following notation is useful for further characterizing conjunctions of prime modal atoms. Let V be a (possibly inconsistent) set of bindings. Define $\phi(V) = \bigwedge\{\beta(v) \mid \langle\beta/v\rangle \in V\}$, where $\beta(\top)$ and $\beta(\bot)$ denote β and $\neg\beta$ resp.

Lemma 5. *$O_i\varphi \wedge \phi(V) \twoheadrightarrow_{R_{\mathsf{Min}}} O_i\varphi\sigma \wedge \phi(V)$ for every substitution σ over V.*

Proof. By induction on the length of σ. The basis step, where $\sigma = \varepsilon$, is trivial. Let σ be some substitution over V, and assume for the induction hypothesis that $O_i\varphi \wedge \phi(V) \twoheadrightarrow_{R_{\mathsf{Min}}} O_i\varphi\sigma \wedge \phi(V)$. Let $\langle\beta/v\rangle \in V$. Then $\phi(V) \wedge \beta(v) \twoheadrightarrow_{R_{\mathsf{Min}}} \phi(V)$ by I$\wedge$ and $\phi(V) \wedge \beta(\overline{v}) \twoheadrightarrow_{R_{\mathsf{Min}}} \bot$ by Kn and Dm. If $\beta \notin \mathrm{pmod}(\varphi\sigma)$, then $\varphi\sigma\langle\beta/v\rangle = \varphi\sigma$. If $\beta \in \mathrm{pmod}(\varphi\sigma)$, then

$$\begin{aligned} O_i\varphi\sigma \wedge \phi(V) &\to_{\mathsf{E}} ((O_i\varphi\sigma\langle\beta/\top\rangle \wedge \beta) \vee (O_i\varphi\sigma\langle\beta/\bot\rangle \wedge \neg\beta)) \wedge \phi(V) \\ &\to_{\mathsf{Dt}} (O_i\varphi\sigma\langle\beta/\top\rangle \wedge \phi(V) \wedge \beta) \vee (O_i\varphi\sigma\langle\beta/\bot\rangle \wedge \phi(V) \wedge \neg\beta) \\ &\twoheadrightarrow_{R_{\mathsf{Min}}} O_i\varphi\sigma\langle\beta/v\rangle \wedge \phi(V) \text{ by the observations above.} \end{aligned}$$

□

If $\alpha \in \mathrm{pmod}(\varphi)$, we may expand $O_i\varphi$ to $e(O_i\varphi, \alpha)$; if in addition $\beta \in \mathrm{pmod}(\varphi\langle\alpha/\top\rangle) \cap \mathrm{pmod}(\varphi\langle\alpha/\bot\rangle)$, we may expand each disjunct once more and apply the distribution rule:

$e(O_i\varphi, \alpha) \twoheadrightarrow_{R_{\mathsf{Min}}} e(O_i\varphi, \alpha, \beta)$. Observe that it may be the case that $\beta \notin \text{pmod}(\varphi)$, and in general it is not the case that $e(O_i\varphi, \alpha, \beta) = e(O_i\varphi, \beta, \alpha)$, nor that $e(O_i\varphi, \alpha) = e(O_i\varphi, \alpha, \alpha)$. What *is* the case, though, is that $e(O_i\varphi, \alpha)$ and $e(O_i\varphi, \beta)$ are R_{Min}-joinable:

Lemma 6. *R_{Min} satisfies permutable expandability.*

Proof. Assume that $\alpha, \beta \in \text{pmod}(\varphi)$, and let $\mu = e(O_i\varphi, \alpha)$ and $\nu = e(O_i\varphi, \beta)$. We show that $\mu \downarrow_{R_{\mathsf{Min}}} \nu$. If $\alpha \neq \beta$, then $\beta \in O_i\varphi\langle\alpha/u\rangle$ and $\alpha \in O_i\varphi\langle\beta/v\rangle$, thus $\mu \twoheadrightarrow_{R_{\mathsf{Min}}} e(O_i\varphi, \alpha, \beta)$ and $\nu \twoheadrightarrow_{R_{\mathsf{Min}}} e(O_i\varphi, \beta, \alpha)$. We have to show that for every u and v, if we let $a = \langle\alpha/u\rangle$ and $b = \langle\beta/v\rangle$, then $O_i\varphi ab \wedge \alpha(u) \wedge \beta(v)$ and $O_i\varphi ba \wedge \alpha(u) \wedge \beta(v)$ are joinable. One can easily construct substitutions σ and τ over $V = \{a, b\}$ – a modal valuation – such that $\varphi ab\sigma$ and $\varphi ba\tau$ are without occurrences of α and β. By Lemma 3, $O_i\varphi ab\sigma = O_i\varphi ba\tau$, and by Lemma 5, $O_i\varphi ba \wedge \phi(V) \twoheadrightarrow_{R_{\mathsf{Min}}} O_i\varphi ba\tau \wedge \phi(V)$ and $O_i\varphi ab \wedge \phi(V) \twoheadrightarrow_{R_{\mathsf{Min}}} O_i\varphi ab\sigma \wedge \phi(V)$. □

4.2. Local Confluence

A rule $l \to r$ satisfies *R-substitutional joinability* if $la \downarrow_{R \cup \{l \to r\}} ra$ whenever l is untainted, for any binding a. We use this property to show joinability when some $O_i\varphi$ is expanded and reduction is performed on some proper subformula of $O_i\varphi$. Some rules satisfy R-substitutional joinability for any R:

- The rules $l \to r$ whose LHS are tainted trivially satisfy the property. If we had allowed nested Os, we would have had to show substitutional joinability for tainted LHS, and this proves much harder than what is presently the case. And as disallowing nested Os does not reduce expressibility, we find it to be worth the sacrifice.
- The rules whose LHS and RHS are objective satisfy the property, as $la = l \to_{R \cup \{l \to r\}} r = ra$ for any R.
- The rules where the only requirement on the variables on the LHS is that they are subjective also satisfy the property, as if $x \in \mathsf{Subj}$, then $x\langle\beta/v\rangle \in \mathsf{Subj}$, hence the rule is still applicable: $la \to_{R \cup \{l \to r\}} ra$.

The only two structural rules that fall outside all three categories are I$\wedge$ and Kn. We say that a rule set R satisfies R'-substitutional joinability if every rule in R does; if $R = R'$ we simply say that R satisfies substitutional joinability. R_{Min} does *not* satisfy substitutional joinability, as neither I$\wedge$ nor Kn satisfy R_{Min}-substitutional joinability.

Lemma 7. *R_{Only} satisfies substitutional joinability.*

Proof. We show R_{Only}-substitutional joinability for I$\wedge$ and Kn. Let $\beta \in \mathsf{PMA}$.

$l \to r$	$l\langle\beta/v\rangle \twoheadrightarrow$	j	$r\langle\beta/v\rangle \twoheadrightarrow$	j
I$\wedge$	$\beta \wedge \beta\langle\beta/v\rangle = v \wedge v \to_x$	v	$\beta\langle\beta/v\rangle =$	v
I$\wedge$	$\neg\beta \wedge \neg\beta\langle\beta/v\rangle = \neg v \wedge \neg v \to_{\mathsf{Co}} \overline{v} \wedge \overline{v} \to_y$	$\overline{v}$	$\neg\beta\langle\beta/v\rangle = \neg v \to_{\mathsf{Co}} \overline{v}$	
Kn	$\beta \wedge \neg\beta\langle\beta/v\rangle = v \wedge \neg v \to_{\mathsf{Co}} \top \wedge \bot \to_z$	$\bot$	$\bot\langle\beta/v\rangle =$	$\bot$

j denotes the common reduct of $l\langle\beta/v\rangle$ and $r\langle\beta/v\rangle$. x, y and z are Dm or J$\wedge$, depending on v. The remaining rules satisfy R-substitutional joinability for any R. □

The following lemma gives us joinability when one of the reducts stems from an application of the expand rule, and the other satisfies substitutional joinability (see Figure 3, left).

Lemma 8. *Let $R \supseteq R_{\mathsf{Min}}$. If $O_i\varphi|_p \to_{l\to r} \omega$ for some rule $l \to r$ satisfying R-substitutional joinability, then $e(O_i\varphi, \alpha) \downarrow_{R\cup\{l\to r\}} O_i\varphi[\omega]_p$ for every $\alpha \in \mathrm{pmod}(\varphi)$.*

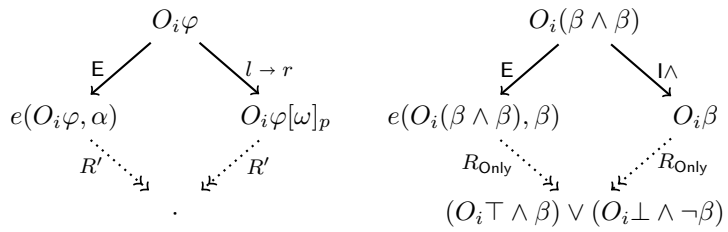


Figure 3. *Left:* Lemma 8: $R' = R \cup \{l \to r\}$-joinability of $e(O_i\varphi, \alpha)$ and $O_i\varphi[\omega]_p$ is guaranteed as long as $l \to r$ satisfies R-substitutional joinability and $R_{\mathsf{Min}} \subseteq R$. *Right*: $e(O_i(\beta\wedge\beta), \beta)$ and $O_i\beta$ are R_{Only}-joinable.

As the expand rule is the source of an infinite number of critical pairs, we have introduced the notion of substitutional joinability in order to show that all critical pairs are joinable. As long as the system contains R_{Min}, it already satisfies permutable expandability.

Theorem 9. *Any rule set R extending R_{Min} is locally confluent if it satisfies substitutional joinability and all critical pairs of $R \setminus \{\mathsf{E}\}$ are joinable.*

Proof. Any critical pair involving the expand rule is joinable by Lemma 6 (as $R_{\mathsf{Min}} \subseteq R$) and Lemma 8 (as $R_{\mathsf{Min}} \subseteq R$ and as long as substitutional joinability holds). Any critical pair not involving the expand rule is joinable by assumption. Hence R is locally confluent by the Critical Pair Lemma. □

5. A Confluent Rule Set Capturing the MRT

In the previous section we gave sufficient properties for a rule set extending R_{Min} to be locally confluent. But what we really want is a rule set that is confluent, not merely locally. By Lemma 4, such a rule set must be locally coherent with AC, which R_{Min} is not, hence we need some additional rules. R_{Min} is not even strong enough to guarantee that there is some normal form that is canonical; the rule set $R_{\mathsf{Comp}} = \{\mathsf{E}, \mathsf{Dt}, \mathsf{C}, \mathsf{Dm}, \mathsf{J}\vee\}$ is, however, as was shown in [21]. Note that R_{Comp} does not extend R_{Min}, but extends a subset of it with the collapse rules. The rule set we are primarily interested in is R_{Only}, comprising the expand rule, all of the structural rules, and the collapse rules, i.e. $R_{\mathsf{Only}} = R_{\mathsf{Min}} \cup \{\mathsf{C}, \mathsf{I}\vee, \mathsf{A2}, \mathsf{A3}, \mathsf{J}\wedge, \mathsf{Co}\}$.

Lemma 10. *Any canonical formula is on normal form wrt. R_{Only}.*

Proof. The only LHS that matches a formula $\mu_1 \vee \cdots \vee \mu_n$ on canonical form is that of the very last collapse rule, i.e. $O_i\varphi \wedge O_j\psi$ for some $i \prec j$. However, its side condition requires that $\{\neg\varphi, \psi\}$ is satisfiable, but as each μ_k is cumulative, this cannot be the case. □

By Lemma 10 and the fact that R_{Only} extends R_{Comp}, R_{Only} also captures the MRT. As R_{Only} extends R_{Min}, we may use Theorem 9 to show that it is confluent. The additional rules in $R_{\mathsf{Only}} \setminus R_{\mathsf{Min}}$ are needed for the following reasons:

- We need the collapse rules to obtain canonical normal forms.
- Associativity of disjunction lets us apply A1 in two distinct ways to some formula (cf. Figure 5), resulting in two distinct reducts which can only be joined by applying A2.
- Now we may apply both A1 and A2 to some formula (cf. Figure 5), resulting in two distinct reducts, joinable by A3 (and A1).
- If we apply A3 to $Op \vee (Op \wedge Bp)$, and $Op \wedge Bp \rightarrow_{\mathsf{C}} Op$, we obtain the two distinct reducts Op and $Op \vee Op$, joinable by I$\vee$.
- J$\wedge$ and Co are needed to show R_{Only}-substitutional joinability.

Although the additional rules are introduced for purely technical reasons (we need A1 to show confluence in a somewhat esoteric case, and because of this, we need A2, and because of this again, we need A3), they are nonetheless useful in practice. Rules structurally similar to the absorption rules are found in [15], where they are called *simplification rules*. In Example 27 of the same paper, all three rules are used when reducing the representation of a default theory with two defaults.

$$
\begin{array}{rll}
O_i\varphi \wedge B_k\psi & \rightarrow_{\mathsf{C}} O_i\varphi & \text{if } i \preccurlyeq k \text{ and not } \mathsf{SAT}(\varphi, \neg\psi) \\
O_i\varphi \wedge B_k\psi & \rightarrow_{\mathsf{C}} \bot & \text{if } k \preccurlyeq i \text{ and } \mathsf{SAT}(\varphi, \neg\psi) \\
O_i\varphi \wedge \neg B_k\psi & \rightarrow_{\mathsf{C}} \bot & \text{if } i \preccurlyeq k \text{ and not } \mathsf{SAT}(\varphi, \neg\psi) \\
O_i\varphi \wedge \neg B_k\psi & \rightarrow_{\mathsf{C}} O_i\varphi & \text{if } k \preccurlyeq i \text{ and } \mathsf{SAT}(\varphi, \neg\psi) \\
\\
O_i\varphi \wedge C_k\psi & \rightarrow_{\mathsf{C}} O_i\varphi & \text{if } k \preccurlyeq i \text{ and not } \mathsf{SAT}(\neg\varphi, \neg\psi) \\
O_i\varphi \wedge C_k\psi & \rightarrow_{\mathsf{C}} \bot & \text{if } i \preccurlyeq k \text{ and } \mathsf{SAT}(\neg\varphi, \neg\psi) \\
O_i\varphi \wedge \neg C_k\psi & \rightarrow_{\mathsf{C}} \bot & \text{if } k \preccurlyeq i \text{ and not } \mathsf{SAT}(\neg\varphi, \neg\psi) \\
O_i\varphi \wedge \neg C_k\psi & \rightarrow_{\mathsf{C}} O_i\varphi & \text{if } i \preccurlyeq k \text{ and } \mathsf{SAT}(\neg\varphi, \neg\psi) \\
\\
O_i\varphi \wedge O_k\psi & \rightarrow_{\mathsf{C}} \bot & \text{if } i \preccurlyeq k \text{ and } \mathsf{SAT}(\neg\varphi, \psi)
\end{array}
$$

Figure 4. The collapse rules. $\varphi, \psi \in \mathsf{Obj}$ and $\mathsf{SAT} : \mathsf{Obj} \times \mathsf{Obj} \Rightarrow \mathsf{Bool}$. We assume that $\mathsf{SAT}(\varphi, \psi)$ reduces to *true* iff $\{\varphi, \psi\}$ is propositionally satisfiable.

In order to obtain confluence, the structural rules are restricted by requiring that variables are of a more specific sort than Fml. Consider $(Op \wedge Bp) \vee (Op \wedge \neg Bp) \rightarrow Op$, which is a typical instance of A1. The reason why we restrict the sort of the variables such that, e.g., $(Bp \wedge Op) \vee (Bp \wedge \neg Op) \rightarrow Bp$ is not an instance of A1, is that $Bp \wedge Op \rightarrow_{\mathsf{C}} Op$, while $Bp \wedge \neg Op$ is on normal form. Hence a less restrictive rule would generate an instance of a critical pair $(Bp, Op \vee (Bp \wedge \neg Op))$ that is not joinable.

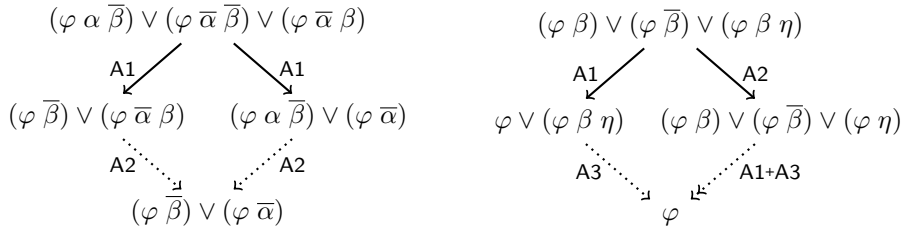


Figure 5. Examples of why the absorption rules A2 and A3 are needed. The conjunction sign $\wedge$ has been omitted to save space. A1 necessitates the inclusion of A2, which again necessitates the inclusion of A3.

Theorem 11. *R_{Only} is confluent.*

Proof. By Lemma 4, R_{Only} is confluent if terminating, locally confluent and locally coherent with AC. By Theorem 9, R_{Only} is locally confluent if it satisfies substitutional joinability and critical pairs (of $R_{\mathsf{Only}} \setminus \{\mathsf{E}\}$) are joinable. Hence we need the following properties. *Joinability of critical pairs:* Left to the reader. *Local coherence with AC:* Left to the reader. *Substitutional joinability:* By Lemma 7. *Termination:* We only give an informal argument. Every rule except the expand and distribution rules decreases the length of the formula. The expand rule is only applicable a finite number of times, as $|\text{mod}(\varphi\langle\beta/v\rangle)| < |\text{mod}(\varphi)|$ if $O_i\varphi \to e(O_i\varphi, \beta)$. No rule increases the number of distinct modal atoms. By itself, the distribution rule can only applied a finite number of times, as it pushes conjunctions inwards. If the distribution rule is not applicable and some other rule than the expand rule is applied, then the distribution rule is still not applicable. Thus any rule is only applicable a finite number of times. $\square$

As mentioned, R_{Comp} captures the MRT, i.e. for any O_I-block φ, there is some canonical μ on normal form such that $\varphi \twoheadrightarrow_{R_{\mathsf{Comp}}} \mu$. Being confluent, R_{Only} has the additional property that if $\varphi \twoheadrightarrow_{R_{\mathsf{Only}}} \mu$ and μ is on normal form, then μ is canonical. In relation to the MRT, this can be viewed as a step towards a correctness result. Correctness requires soundness, a property which relies on a formal semantics. To establish soundness one must, relative to an appropriate notion of validity, show that the LHS of the rules are logically equivalent to the resp. RHS, and that substitution of logical equivalents is truth preserving. These properties have been established for R_{Comp} wrt. the formal semantics of the only-knowing logics in [21]. Since the rules in R_{Only} that are not in R_{Comp}, are based on propositional tautologies, R_{Only} is, in the sense of the term just described, correct.

6. Conclusion and Future Work

We have in this paper proved termination and confluence of a rewrite system for one of the most basic only-knowing logics [13,21]. The proof is generic in the sense that it is based on the notion of substitutional joinability, a concept which can be used to show confluence also in cases where one wants to add new rules to the system. Using the rewriting logic tool Maude [3], we are currently experimenting with an implementation of the system, investigating the effect of adding further simplification rules and changing the search strategy.

References

[1] Grigoris Antoniou. *Nonmonotonic Reasoning*. The MIT Press, Cambridge, Massachusetts, 1997.

[2] Franz Baader and Tobias Nipkow. *Term Rewriting and All That*. Cambridge University Press, 1998.

[3] Manuel Clavel, Francisco Durán, Steven Eker, Patrick Lincoln, Narciso Martí-Oliet, José Meseguer, and José F. Quesada. Maude: Specification and Programming in Rewriting Logic. *Theoretical Computer Science*, 285:187–243, August 2002.

[4] Nachum Dershowitz and David A. Plaisted. Rewriting. In *Handbook of Automated Reasoning*, pages 535–610. 2001.

[5] Jürgen Dix, Ulrich Furbach, and Ilkka Niemelä. Nonmonotonic reasoning: Towards efficient calculi and implementations. In *Handbook of Automated Reasoning*, pages 1241–1354. 2001.

[6] Iselin Engan, Tore Langholm, Espen H. Lian, and Arild Waaler. Default Reasoning with Preference Within Only Knowing Logic. In *Proceedings of LPNMR 2005*.

[7] Georg Gottlob. Complexity Results for Nonmonotonic Logics. *J. Log. Comput.*, 2(3):397–425, 1992.

[8] Joseph Y. Halpern and Gerhard Lakemeyer. Multi-agent only knowing. *J. Log. Comput.*, 11(1):41–70, 2001.

[9] Gérard P. Huet. Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems. *J. ACM*, 27(4):797–821, 1980.

[10] Jean-Pierre Jouannaud and Hélène Kirchner. Completion of a Set of Rules Modulo a Set of Equations. *SIAM J. Comput.*, 15(4):1155–1194, 1986.

[11] Gerhard Lakemeyer and Hector J. Levesque. Only-Knowing: Taking It Beyond Autoepistemic Reasoning. In *Proceedings of AAAI 2005*.

[12] Gerhard Lakemeyer and Hector J. Levesque. Towards an Axiom System for Default Logic. In *Proceedings of AAAI 2006*.

[13] Hector J. Levesque. All I Know: A Study in Autoepistemic Logic. *Artif. Intell.*, 42(2-3):263–309, 1990.

[14] Hector J. Levesque and Gerhard Lakemeyer. *The Logic of Knowledge Bases*. MIT Press, 2001.

[15] Espen H. Lian and Arild Waaler. Computing Default Extensions by Reductions on O^R. In *Proceedings of KR 2008*, pages 496–506. AAAI Press.

[16] Espen H. Lian and Arild Waaler. An Only Knowing Approach to Defeasible Description Logics (extended abstract). In *DL 2009*, 2009.

[17] José Meseguer. Conditional Rewriting Logic as a United Model of Concurrency. *Theoretical Computer Science*, 96(1):73–155, 1992.

[18] Enno Ohlebusch. *Advanced Topics in Term Rewriting*. Springer Verlag, 2002.

[19] Riccardo Rosati. A sound and complete tableau calculus for reasoning about only knowing and knowing at most. *Studia Logica*, 69(1):171–191, 2001.

[20] Arild Waaler. Consistency proofs for systems of multi-agent only knowing. In *Proceedings of AiML 2004*.

[21] Arild Waaler, Johan W. Klüwer, Tore Langholm, and Espen H. Lian. Only knowing with degrees of confidence. *J. Applied Logic*, 5(3):492–518, 2007.

[22] Arild Waaler and Bjørnar Solhaug. Semantics for multi-agent only knowing: extended abstract. In Ron van der Meyden, editor, *Proceedings of the 10th Conference on Theoretical Aspects of Rationality and Knowledge (TARK)*, pages 109–125, 2005.